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# Report 3

# Kalman or Wiener Filters

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# 1 OVERVIEW

Everywhere in the real world we are surrounded by signals like temperature, sound, voltage and so on. To extract useful information, to further process or analyse them, it is necessary to use filters.

## 1.1 SPECTRAL FILTERS

The term “filter” traditionally refers to **spectral filters** with **constant coefficients** like “Finite Impulse Response” (FIR) or “Infinite Impulse Response” (IIR). To apply such filters, a **constant, periodic signal** is converted from time domain to frequency domain, in order to analyse its spectral attributes and filter it as required, typically using fixed Low-Pass, Band-Pass or High-Pass filters.

Fig. 1 shows this process on the example of two signals – with the original signals in blue colour and the signals after filtering in green colour. The blue graph in the top left shows a simple signal consisting of three sine waves with frequencies of 1.000, 2.000 and 3.000 Hz. Using Fourier Theorem, this signal is converted from time to frequency domain, as shown in blue in the graph in the top right. The target is to remove the signals at 2.000 and 3.000 Hz from the signal, so a Low-Pass filter like then one shown in Fig. 2 is applied, with the results being shown in green. The resulting signal after applying the filter is shown in green in the top left graph. The same experiment is then repeated for an audio signal in the second row.

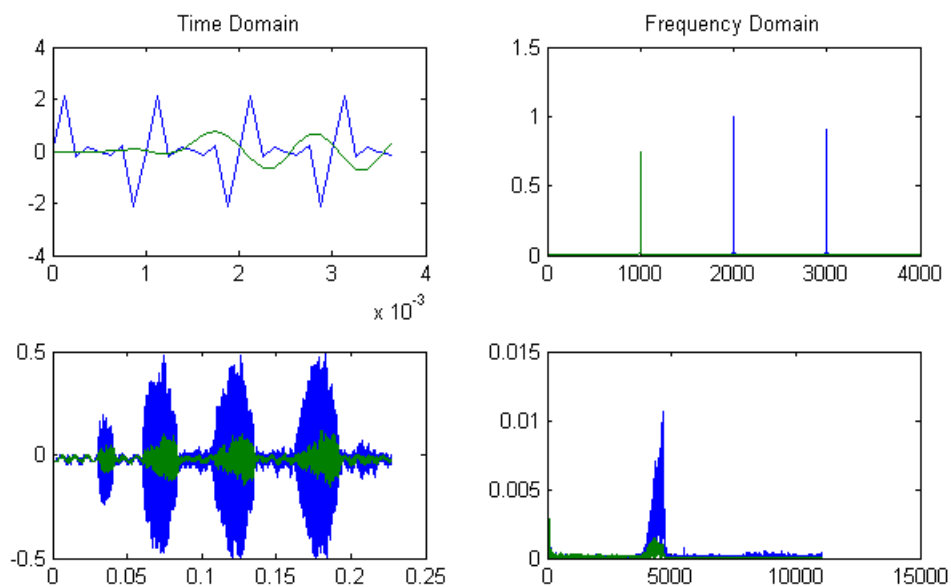


Fig. 1 Example signals before (blue) and after (green) applying a Low-Pass FIR-Filter

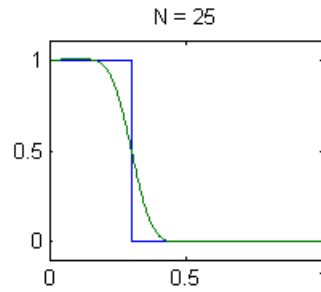


Fig. 2 Frequency characteristics of a Low-Pass Filter

This approach works well, if the limits of the noise spectrum is outside the signals spectrum. The problem arises, when the noise spectrum overlaps the signals spectrum, as it becomes impossible to remove the noise using spectral analyzing. In that case a new approach is needed: adaptive filters.

## 1.2 ADAPTIVE FILTERS

In real applications we are very often in the situation, that signal and noise spectra are overlapping – as can be seen in Fig. 3.

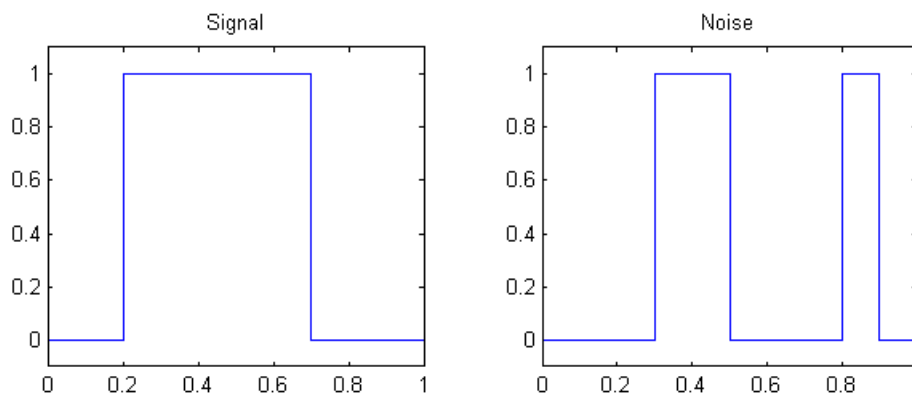


Fig. 3 Signal and Noise overlapping in frequency domain

For example whenever recording sound with a microphone, not only the purposed signal is recorded, but also certain unwanted sounds, like background sounds. A certain unwanted sound, that is practically always appearing, and where adaptive filters are needed, is **white noise**. Fig. 4 shows how such a noise is added to the original signal when measuring.

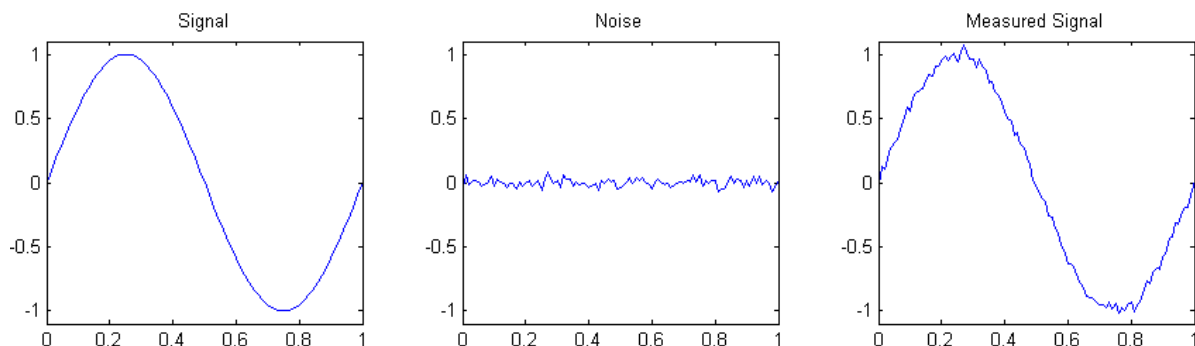


Fig. 4 Original Signal + White Noise -> Measured SignalA

White noise is defined as a random, non-deterministic signal with a constant mean ( $\mu$ ) and variance ( $\sigma^2$ ) for a large number of  $x$ . It is fully described by the “**Probability Density Function**” (PDF):

$$f(x) = \frac{1}{\sigma \times \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Eq. 1 Probability Density Function (PDF)

This means, that the noise signal is distributed according to a **Gaussian bell distribution** (Fig. 5).

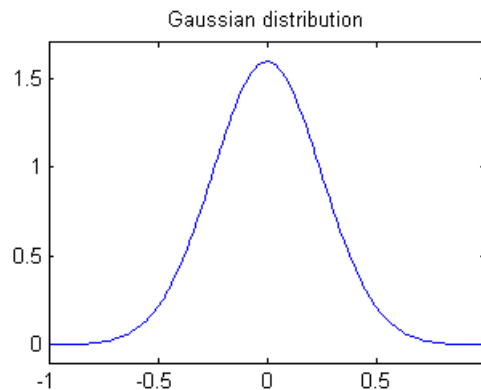


Fig. 5 Gaussian distribution

The **statistical values** of mean and variance can be extracted from a signal using the following formulas:

$$\mu = \frac{1}{N} \sum_{k=1}^N x_{[k]}$$

Eq. 2 Extracting mean value

$$\sigma^2 = \frac{1}{N} \sum_{k=1}^N (x_{[k]}^2 - \mu^2)$$

Eq. 3 Extracting variance

For problems like this, where the spectrum of signal and noise are overlapping and as such spectral filters are not working, **adaptive filters** try to exploit those statistical values to remove the noise in **time domain**. The algorithm is **linear**, similar to FIR filters, and tries to minimize the difference between a corrupted signal (“measured signal”) and a desired signal using the “Least Mean Squared” (LMS) or “Recursive Least Squared” (RLS) error.

## 2 WIENER-FILTER

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*“In signal processing, the Wiener filter is a filter used to produce an estimate of a desired or target random process by linear time-invariant filtering of an observed noisy process, assuming known stationary signal and noise spectra, and additive noise. The Wiener filter minimizes the mean square error between the estimated random process and the desired process.” [1]*

### 2.1 HISTORY

In the 1940s the American mathematician Norbert Wiener constructed the theory of filtering stationary time series. The paper [2] was published in 1949.

The theory of Wiener filters is based on causal filters that operate only on past and present time series and try to predict an unavailable future. It was developed during the Second World War in the need for determining where to aim anti-aircraft guns at dodging planes. An analogue computer in the gunsight tracked the moving airplane, thus generating a time history of its movements and some autocovariances. It then forecasted the optimal aiming point for the gun, in order to hit the position of the other airplane with the smallest deviation. [3]

In 1941 a similar algorithm was published by the soviet mathematician Andrey Kolmogorov. Hence the theory is often called “Wiener–Kolmogorov” filter.

### 2.2 DEFINITION

The system of Wiener filters is visualized in Fig. 6. The idea is to apply a linear filter with coefficients  $w_k$  on the measured signal, which is an aggregation of the desired signal ( $d_{[n]}$ ) and noise ( $v_{[n]}$ ). The result ( $y_{[n]}$ ) is used to calculate the error ( $\epsilon$ ) in reference to the desired signal ( $d_{[n]}$ ). The adaptive algorithm then tries to adjust the coefficients ( $w_k$ ) to minimize this error ( $\epsilon = y_{[n]} - d_{[n]} \rightarrow 0$ ).

As Wiener filters expect the statistical values (mean and variance) of the error to be constant, searching for the optimal coefficients is done only once. The actual application of the filter in a real system uses those constant coefficients.

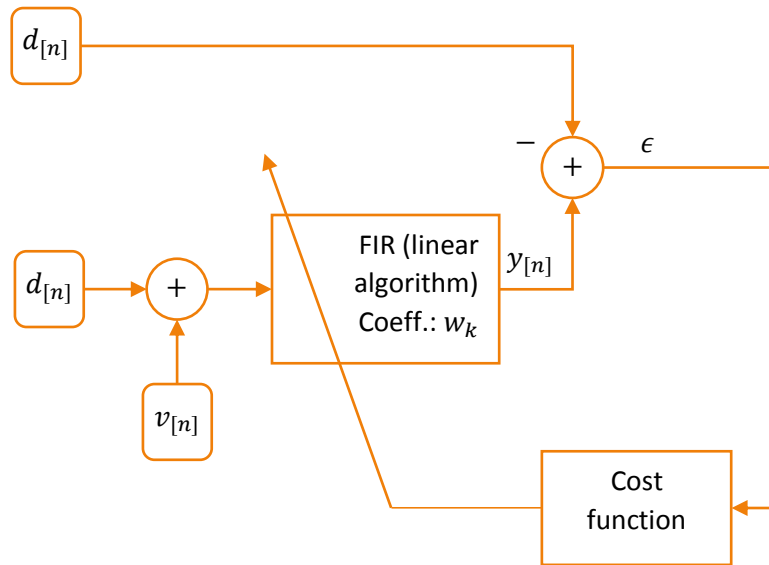


Fig. 6 Schema for Wiener filter

The equation to apply the filter to calculate  $y[n]$  is as follows:

$$y[n] = \sum_{k=0}^{N-1} w_k \times (d_{[n-k]} + v_{[n-k]})$$

Eq. 4 Applying the filter

The core function of Wiener filters is the Wiener-Hopf-Equation, which is used to calculate the optimal coefficients ( $W_{opt}$ ):

$$R \times W_{opt} = P$$

Eq. 5 Wiener-Hopf-Equation

In this equation,  $R$  is the autocorrelation matrix of the input,  $P$  is an intercorrelation matrix between input and desired signal and  $W_{opt}$  are the optimal filter coefficients that produce  $\epsilon \rightarrow 0$ .

## 2.3 EXAMPLE

Considering again the following definitions: [4]

- Measured signal:  $X$ ;  $x[n]$
- Desired signal:  $D$ ;  $d[n]$
- Noise:  $v[n]$
- Coefficients:  $W$ ;  $w_k$
- Error:  $\epsilon$

From  $\epsilon \rightarrow 0$  we expect the product of the minimum error with the measured signal to be 0. Also the difference between desired signal and the product between the optimal coefficients and the measured signal is 0:

$$E[\epsilon_{min} X_{[n]}] = E[D - W_{opt} X] = 0$$

Eq. 6 Expectation 1

From this we can derive and define as follows:

$$E[X \times X^T] W_{opt} = E[X \times D]$$

$$R_x = E[X \times X^T] = \sum_{i=0}^{L-1} E[x_{[n-k]} \times x_{[n-i]}]; \text{ for } k = 0, 1, 2, \dots$$

$$r_{xd} = E[X \times d]$$

Eq. 7 Expectation 2

The autocorrelation function  $R_x$  of the filter input for a lag of  $i - k$  can be expressed as:

$$r(i - k) = E[x_{[n-k]} \times x_{[n-i]}]$$

Eq. 8 Expectation 3

In the same way, the intercorrelation between filter input  $x_{[n-k]}$  and the desired signal  $D$  for a lag of  $-k$ , can be expressed as:

$$p(-k) = E[x_{[n-k]} \times d_{[n]}]$$

Eq. 9 Expectation 4

Combining these expectations according to Wiener-Hopf-Equation (Eq. 5), we receive a system of equations as the condition for the optimality of the filter:

$$\sum_{i=0}^{L-1} w_{opt} \times r_{[i-k]} = p_{[-k]}; \text{ for } k = 0, 1, 2, \dots$$

Eq. 10 System of equations

To match the definition of Wiener-Hopf-Equation in (Eq. 5), we reform the expectations into a matrices:

$$R = \begin{bmatrix} r_{[0]} & r_{[1]} & \dots & r_{[L-1]} \\ r_{[1]} & r_{[0]} & & r_{[L-2]} \\ \vdots & & \ddots & \\ r_{[L-1]} & r_{[L-2]} & & r_{[0]} \end{bmatrix}$$

$$P = \begin{bmatrix} p_{[0]} \\ p_{[-1]} \\ \vdots \\ p_{[1-L]} \end{bmatrix}$$

Eq. 11 Building matrices for Wiener-Hopf-Equation

Assuming that  $R$  is a regular matrix with a possible inverse matrix, we just need to solve this equation, to receive the optimal filter coefficients:

$$W_{opt} = R^{-1} \times P$$

Eq. 12 Wiener-Hopf-Equation for  $W_{opt}$



## 3 KALMAN-FILTER

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*Kalman filtering, also known as linear quadratic estimation (LQE), is an algorithm that uses a series of measurements observed over time, containing noise (random variations) and other inaccuracies, and produces estimates of unknown variables that tend to be more precise than those based on a single measurement alone. More formally, the Kalman filter operates recursively on streams of noisy input data to produce a statistically optimal estimate of the underlying system state. The filter is named after Rudolf (Rudy) E. Kálmán, one of the primary developers of its theory. [5]*

### 3.1 HISTORY

In 1960 the Hungarian born American mathematician Rudolf (Rudy) Emil Kálmán extended Wiener filters to non-stationary, dynamic processes in his paper “A New Approach to Linear Filtering and Prediction Problems”. [6]

The need for this improvement was again arising from a military problem. The stationary forecasting of Wiener filters was not fitting to the problem of prediction of the trajectories of ballistic missiles, because they showed very different characters during launch and re-entry phase.

The first widely recognized use of Kalman filters was during the Apollo 11 program, when landing on Moon in 1969. Nowadays they are widely used in the “Inertial navigation system” that is used for short-term navigation in airplanes, but also in tracking methods, for autonomic vehicles to reduce noise of radar devices, and many more.

### 3.2 DEFINITION

In difference to traditional filters like FIR and IIR, the Kalman filter has a more complex structure. It internally makes use of the state-space model, which allows it to handle dynamic models with varying parameters. The filter is implemented as a recursive method, as it reuses previous outputs as inputs. This makes it very efficient and convenient to implement as a computer algorithm.

The following parameters are used:

- Measured signal:  $x_k$
- Desired signal:  $d_k$
- Kalman Gain:  $K_k$
- Estimated desired signal:  $\hat{d}_k$
- Control signal:  $u_k$
- Process noise:  $w_k$
- Measurement Noise:  $v_k$
- State-Transition model:  $A_k$
- Control-input model:  $B_k$
- Observation model:  $H_k$   
(Matrices  $A_k$ ,  $B_k$  and  $H_k$  can be expected to be numerical constants in most cases, most probably even equal to 1)
- Covariance of process noise:  $Q_k$

- Covariance of observation noise:  $R_k$
- Error covariance:  $P_k$

The process is generally split in two parts: “Time Update (prediction)”, which projects the state and the error covariance, and “Measurement Update (correction)”, which calculates the Kalman Gain to correct those projections. [7]

To start the process, estimates for  $d_0$  and  $P_0$  need to be provided to the algorithm.

### **Time Update (Prediction)**

1. Project (predict) the next state:

$$\hat{d}_k = A \times \hat{d}_{k-1} + B \times u_k$$

*Eq. 13 Step 1: Project state*

2. Project the error covariance ahead:

$$P_k = A \times P_{k-1} \times A^T + Q$$

*Eq. 14 Step 2: Project error covariance*

### **Measurement Update (Correction)**

3. Compute the Kalman Gain:

$$K_k = P_k \times H^T \times (H \times P_k \times H^T + R)^{-1}$$

*Eq. 15 Step 3: Compute the Kalman Gain*

4. Update the projected state via  $x_k$

$$\hat{d}_k = \hat{d}_k + K_k \times (x_k - H \times \hat{d}_k)$$

*Eq. 16 Step 4: Update projected state*

5. Update the projected error covariance

$$P_k = (I - K_k \times H) \times P_k$$

*Eq. 17 Step 5: Update projected error covariance*

6. Reiterate the process, using outputs  $k$  as input for  $k + 1$ .  $\hat{d}_k$  is holding the predicted value, the desired result of the Kalman filter.

## **3.3 EXAMPLE**

The tool “Kalman\_Const\_Single” (Fig. 7) by Professor MiHu, shows how Kalman filter is iteratively adjusting to Gaussian process and signal noises. After a few iterations of its algorithm, it is very converging to the constant signal, filtering the measure noise, as is reflected by the statistical values of a stable mean (Mu) and a low variance (Sigma).

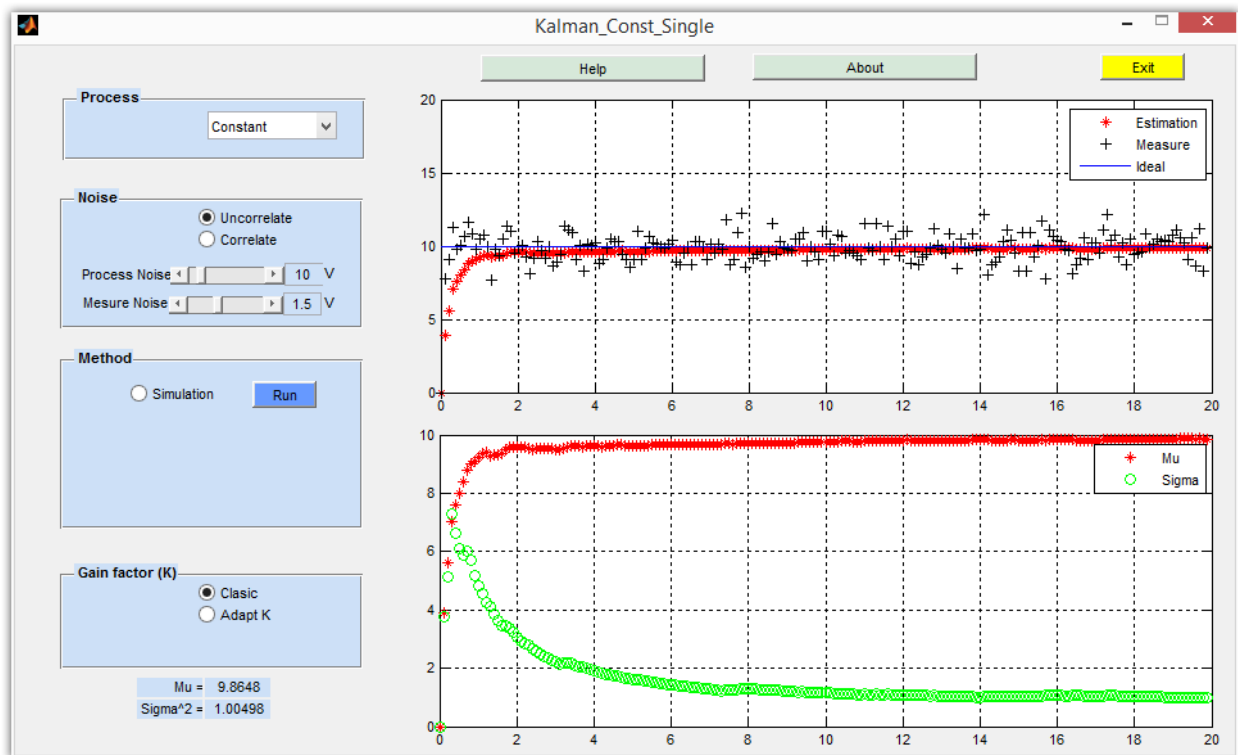


Fig. 7 The tool Kalman\_Const\_Single

## 4 CONCLUSIONS

While it is evident, that both Wiener and Kalman filters are working properly, if fed with the current parameters, there are some improvements in Kalman filters, which make it most likely the perfect choice for most applications.

One obvious implementation problem with Wiener filters is, that the noise signal needs to be known before the actual process, to be able to calculate the optimal coefficients. In many cases this will result in a less optimal filter in the process, as the noise might differ significantly from what was expected. Also – and this was also the reason for Rudolf Kalman to develop “his” filter in the first place – the fact, that real world signals and noises are usually not constant, demands for a self-adjusting filter instead of a static one. Furthermore, the iterative structure of Kalman filter makes it easy and highly efficient to be implemented as a computer algorithm or even executed in a hardware DSP (Digital Signal Processor).

Nevertheless, both filter methods are still a topic in recent scientific papers and continue to be improved and fitted to new use-cases.

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